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MORPHISMS OF LOCAL DYNAMICAL SYSTEMS, I

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Morphisms of local dynamical systems, I*)

by

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ABSTRACT

In this paper we deal with the definition of morphisms between local dynamical systems and their basic properties. Our definition is such that the corresponding isomorphisms are essentially phase space homeomorphisms with reparametrization. We discuss the preservation properties of morphisms with respect to several dynamical properties. Finally, the close relationship between phase space homeomorphisms with reparametrization on the one hand and geometric equivalences on the other, as exhibited by work of URA and KIMURA, is extended to a characterization of morphisms.

KEYWORDS & PHRASES: *local dynamical system, reparametrization, equivariant mapping, geometric equivalence, morphism, limit set, stability, recurrence.*

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*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

This paper deals with the definition of mappings between local dynamical systems and their basic properties. At a first glance there are many ways of defining mappings between dynamical systems [5] and the final choice largely depends on the selection of the dynamical properties whose invariance under mappings is required. As the notion of isomorphism arises in a natural way from that of a mapping between local dynamical systems, the choice of the isomorphisms is also a decisive factor.

Apart from some minor modifications, such as preservation of orientation, the following isomorphism types can be found in the literature: a phase space homeomorphism preserving orbits, also called geometric equivalence [7]; a phase space homeomorphism with reparametrization [10], including equivariant (i.e. without time change) homeomorphisms [4] as a special case and related to homomorphisms of topological transformation groups; and the so-called conjugacy relation [2]: a reparametrization followed by an equivariant homeomorphism.

In section 3 we introduce our choice of a mapping or morphism of local dynamical systems. The isomorphisms of the resulting category are essentially the phase space homeomorphisms with reparametrization. The condition (3.1.2) in the definition of morphism looks rather technical, but, as is also illustrated by examples like (4.4), it turns out to be the key to a satisfactory discussion of the preservation of the various dynamical properties. See section 4 for more details. There is also a close relationship between the isomorphisms as defined in this paper and the conjugacy relation (3.5). Omitting some minor details a more general result can be stated as follows: a morphism is the composition of a reparametrization and an equivariant continuous phase space map. The discussion of this result will appear elsewhere [1].

The close relationship between phase space homeomorphisms with reparametrization and the geometric equivalences, which has been exhibited by URA [9] and KIMURA [6], is extended to a characterization of morphisms in section 5. Roughly speaking, morphisms are orbit preserving continuous maps between phase spaces. The proof follows the pattern of that in [10], resolving the difficulties involving separation axioms by applying a lemma of J. and M. LEWIN, instead of following the approach in [6].

2. PRELIMINARIES

2.1. Throughout the paper we have the following conventions. All spaces are supposed to be Hausdorff spaces. $\mathbb{R}$ denotes the set of real numbers with its usual topology and algebraic operations. $\mathbb{R}^+ := \{t \in \mathbb{R} : t \geq 0\}$ and $\mathbb{R}^- := \{t \in \mathbb{R} : t \leq 0\}$ (expressions like $P := Q$ or $Q =: P$ denote that P is defined to be Q). For $A \subset \mathbb{R}$ we let $A^+ := A \cap \mathbb{R}^+$ and $A^- := A \cap \mathbb{R}^-$.
If X,Y,Z and U are sets, $Z \subset X \times Y$ and if $\pi: Z \to U$ is a mapping, then for all $(x,y) \in Z$ we define

$$\pi_x(y) := \pi(x,y) =: \pi^y(x).$$

2.2. For the convenience of the reader we collect some definitions, including that of a local dynamical system. The various definitions of local dynamical systems [4],[8],[10],[11] are mutually equivalent, although they look somewhat different. See [4], in particular chapters IV and VI, [8] and [10] for some details.

Let X be a topological space, $D \subset X \times \mathbb{R}$ and let $\pi: D \to X$ be a mapping. The triple (X,D,π) is called *a local dynamical system on* X if the following conditions are satisfied:

2.2.0. The set D is open in the product topology of $X \times \mathbb{R}$ and has the following special form:

$$D = \cup\{\{x\} \times J(x) : x \in X\},$$

where J(x) is an (open) interval in $\mathbb{R}$ containing 0 for every $x \in X$.

2.2.1. *(Continuity)*. The mapping $\pi: D \to X$ is continuous.

2.2.2. *(Identity)*. If $x \in X$, then $\pi(x,0) = x$.

2.2.3. *(Group axiom)*. If t, $t+s \in J(x)$ and if $s \in J(\pi(x,t))$, then $\pi(\pi(x,t),s) = \pi(x,t+s)$.

2.2.4. *(Maximality)*. $J(\pi(x,t)) = J(x)-t$ for all $(x,t) \in D$.

The space X and the mapping π are called the *phase space* and *phase mapping* respectively. In case $D = X\times\mathbb{R}$, (X,D,π) is called a *global dynamical system*. For every $x \in X$ the set $\Gamma(x) := \pi_x(J(x))$ is called *the orbit of* x and the sets $\Gamma^+(x) := \pi_x(J^+(x))$ and $\Gamma^-(x) := \pi_x(J^-(x))$ are called *the positive* and *negative semiorbits of* x, respectively. For every $x \in X$ the interval

$J(x)$ is of the form $(\alpha(x),\omega(x))$ with $-\infty \leq \alpha(x) < 0 < \omega(x) \leq \infty$ and the (possibly empty) sets

$$A(x) := \bigcap\{\mathrm{cl}\ \pi_x(\alpha(x),t] \ \vdots\ \alpha(x) < t \leq 0\} \text{ and}$$
$$\Omega(x) := \bigcap\{\mathrm{cl}\ \pi_x[t,\omega(x)) \ \vdots\ 0 \leq t < \omega(x)\}$$

are called the *negative* and *positive limit sets of* x respectively.

When dealing with sets in various local dynamical systems, we shall use the symbol for the phase mapping as a subscript to indicate the system in which the sets are considered. The orbit of x in the system with phase map π for example is denoted by $\Gamma_\pi(x)$.

2.3. To facilitate the further exposition and to motivate certain choices we shall discuss three examples here.

2.3.1. Let X be an open subset of $\mathbb{R}^n$ and let $f\colon X \to \mathbb{R}^n$ be continuous and such that the autonomous differential equation $\dot{x} = f(x)$ has unique solutions. Let $\pi_x : J(x) \to X$ denote the unique solution of $\dot{x} = f(x)$ satisfying the initial condition $\pi_x(0) = x$; here $J(x)$ denotes the maximal open interval on which the solution is defined. If $D := \bigcup\{\{x\} \times J(x) \ \vdots\ x \in X\}$ and $\pi(x,t) := \pi_x(t)$ for $(x,t) \in D$, then (X,D,π) is a local dynamical system (cf. [8], III A).

2.3.2. Each topological space X can be considered as the phase space of at least one global dynamical system, namely the *trivial system* $(X, X \times \mathbb{R}, \pi)$ where π is defined by $\pi(x,t) := x$. A countable space admits no other local dynamical system (cf. [4], II. 1.7 and II. 4.5).

2.3.3. Let (X,D,π) be a local dynamical system. Let $D_* := \{(x,t) \in X \times \mathbb{R} \vdots (x,-t) \in D\}$ and $\pi_*(x,t) := \pi(x,-t)$ for $(x,t) \in D_*$. Then it is easily seen that (X,D_*,π_*) is a local dynamical system. It is called the *reverse system of* (X,D,π) (cf. [4], II. 1.9 and II. 3.10).

It should be observed that for every $x \in X$ we have $\Gamma_\pi(x) = \Gamma_{\pi_*}(x)$ and, for example, $A_\pi(x) = \Omega_{\pi_*}(x)$. This allows us to restrict the discussion to results about positive semiorbits, positive limit sets, etc.

3. MORPHISMS

3.1. We now define morphisms of local dynamical systems. A *morphism (of local dynamical systems)* Φ *from* (X,D,π) *to* (Y,E,ρ) is a pair (φ,τ) satisfying the following conditions:

3.1.1. φ: $X \to Y$ and τ: $D \to \mathbb{R}$ are continuous mappings.

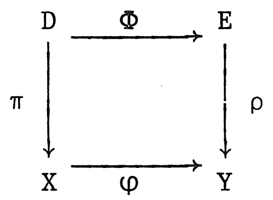
3.1.2. For every $x \in X$ the mappping τ_x: $J_\pi(x) \to \mathbb{R}$ (see 2.1) is strictly increasing and satisfies $\tau_x(0) = 0$ and $\tau_x(J_\pi(x)) \subset J_\rho(\varphi(x))$.

3.1.3. $\varphi(\pi(x,t)) = \rho(\varphi(x),\tau(x,t))$ for every $(x,t) \in D$.

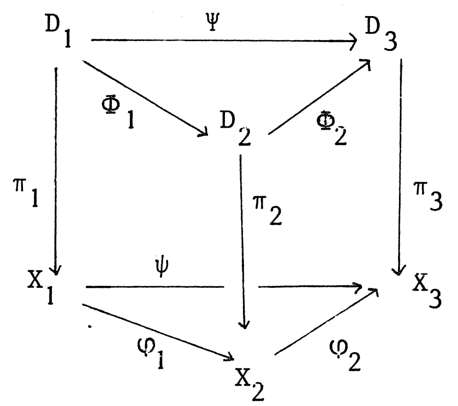
Notation: Φ: $(X,D,\pi) \to (Y,E,\rho)$ or (φ,τ): $(X,D,\pi) \to (Y,E,\rho)$.

A morphism of the form $(1_X,\tau)$: $(X,D,\pi) \to (X,E,\rho)$, where 1_X is the identity map of X, is called a *parameter transformation* and τ is called a *reparametrization* from π to ρ. If (φ,ι) is a morphism and $\iota(x,t) = t$ for all (x,t), then φ is also called an *equivariant mapping*. Morphisms which are of this type will consistently be denoted by (φ,ι) and will sometimes be called *equivariant morphisms* (these are just the morphisms which are studied in [4]).

3.2. If $\Phi := (\varphi,\tau)$: $(X,D,\pi) \to (Y,E,\rho)$ is a morphism, the symbol Φ will also be used to denote the mapping Φ: $D \to E$ defined by $\Phi(x,t) := (\varphi(x),\tau(x,t))$. Observe that in view of the condition 3.1.2 this makes sense. The condition 3.1.3 is equivalent to the commutativity of the following diagram



Thus, a morphism may also be viewed as a special mapping Φ: $D \to E$. From this viewpoint the following definition of the composition of morphisms is quite natural. Let for $i = 1,2$ the morphisms $\Phi_i = (\varphi_i,\tau_i):(X_i,D_i,\pi_i) \to (X_{i+1},D_{i+1},\pi_{i+1})$ be given. The *composition* Ψ *of* Φ_1 *and* Φ_2 is defined by $\Psi := \Phi_2 \circ \Phi_1$. Observe that, writing $\Psi = (\psi,\sigma)$, where $\psi := \varphi_2 \circ \varphi_1$ and $\sigma(x,t) := \tau_2(\Phi_1(x,t)) = \tau_2(\varphi_1(x),\tau_1(x,t))$, we have commutativity of the following diagram



Since $\psi_x = \tau_{2\varphi_1(x)} \circ \tau_{1x}$ as is easily seen, $\Psi = (\psi,\tau)$ actually satisfies the conditions in 3.1.

In this way we now have defined a category whose objects are the local dynamical systems. The isomorphisms in this category will be called *isomorphisms of local dynamical systems*. In the following proposition a more explicit description of the isomorphisms is given.

3.3. PROPOSITION. *Let* $\Phi = (\varphi,\tau)$ *be a morphism from* (X,D,π) *to* (Y,E,ρ). *The following conditions are equivalent:*

(i) Φ *is an isomorphism,*

(ii) Φ: $D \to E$ *is a homeomorphism,*

(iii) φ: $X \to Y$ *is a homeomorphism and for every* $x \in X$ *the mapping* τ_x: $J_\pi(x) \to J_\rho(\varphi(x))$ *is surjective.*

In case that (φ,τ) *is an isomorphism, its inverse* (ψ,σ) *is given by* $\psi := \varphi^{-1}$ *and* $\sigma_y := \tau^{-1}_{\psi(y)}$ *for every* $y \in Y$.

PROOF. Φ is an isomorphism iff it has a two-sided inverse $\Psi = (\psi,\sigma)$. That is, $(\psi,\sigma)\circ(\varphi,\tau)$ and $(\varphi,\tau)\circ(\psi,\sigma)$ are the identity morphisms of (X,D,π) and (Y,E,ρ) respectively. The description of the inverse and the proof of the implication (i) $\Rightarrow$ (iii) now easily follow. The implication (iii) $\Rightarrow$ (ii) is lemma 1 in [3] (observe that, since (φ,τ) is a morphism, the map τ is continuous). Finally we verify the implication (ii) $\Rightarrow$ (i). As (φ,τ) is a morphism, the inverse Ψ: $E \to D$ of the homeomorphism Φ: $D \to E$ is necessarily of the form Ψ: $(y,u) \mapsto (\psi(y),\sigma(y,u))$ with continuous functions ψ: $Y \to X$ and σ: $E \to \mathbb{R}$. By straightforward computation it follows that (ψ,σ) is a morphism and that it is the two-sided inverse of (φ,τ). □

3.4. The following examples and remarks may illustrate the above discussion.

3.4.1. Let in the complex plane $\mathbb{C}$ the global dynamical systems $(\mathbb{C},\mathbb{C}\times\mathbb{R},\pi)$ and $(\mathbb{C},\mathbb{C}\times\mathbb{R},\rho)$ be defined by $\pi(z,t) := z+it$ and $\rho(z,t) := ze^{2it}$ respectively. A morphism (φ,τ) of the system $(\mathbb{C},\mathbb{C}\times\mathbb{R},\pi)$ of translations to the system $(\mathbb{C},\mathbb{C}\times\mathbb{R},\rho)$ of rotations is given by $\varphi(z) := e^z$ and $\tau(z,t) := \frac{t}{2}$.

3.4.2. In the situation of example 2.3.1 let h: $X \to \mathbb{R}$ be continuous and $h(x) > 0$ for every $x \in X$. Then the differential equation $\dot{x} = h(x)f(x)$ has also unique solutions. Let in a similar fashion as in example 2.3.1 the local dynamical system (X,E,ρ) be defined for this equation. Then there exists a reparametrization τ from π to ρ, which, as is well-known from the theory of differential equations, is given by

$$\tau(x,t) := \int_0^t h(\pi(x,s))^{-1} ds.$$

It turns out that $(1_X,\tau)$ is an isomorphism.

It can be shown that every reparametrization is an isomorphism, provided that the set of rest points of the system is nowhere dense. In general the reparametrization can be adjusted on the set of rest points in such a way that it becomes an isomorphism. See [1] for more details. It has been proved by CARLSON [3] and URA & EGAWA [12] that every local dynamical system on a countably paracompact normal space is isomorphic to a global dynamical system under a reparametrization.

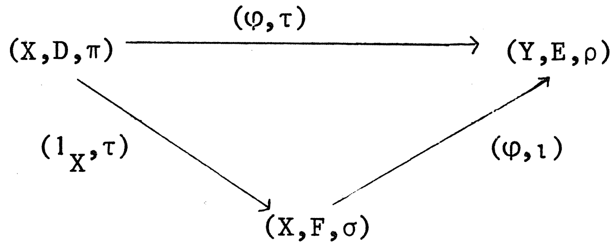
3.4.3. Let (Y,E,ρ) be a local dynamical system and let X be an open subset of Y. If for every $x \in X$, $J_\pi(x)$ denotes the maximal subinterval of $J_\rho(x)$, which contains 0 and for which $\rho(\{x\} \times J_\pi(x)) \subset X$, and if we write $D := \cup\{\{x\} \times J_\pi(x) \vdots x \in X\}$ and $\pi := \rho|D$, then (X,D,π) is a local dynamical system (see [9] or [10]). Then, if $\varphi: X \to Y$ is the embedding mapping and $\iota(x,t):=t$ for every $(x,t)\in D$, we have an equivariant morphism $(\varphi,\iota):(X,D,\pi)\to(Y,E,\rho)$.

Observe that, if X is a non-invariant open subset of Y, then for some $x \in X$ the interval $J_\pi(x)$ is a proper subset of $J_\rho(x)$.

3.4.4. Let $(\varphi,\tau):(X,D,\pi)\to(Y,E,\rho)$ be a morphism. Writing $\tau_*(x,t):= -\tau(x,-t)$, we get a morphism $(\varphi,\tau_*):(X,D_*,\pi_*)\to(Y,E_*,\rho_*)$ of the corresponding reverse systems. This gives rise to a covariant functor of the category of local dynamical systems into itself (cf. [4], p.61).

The following proposition shows how the isomorphisms defined above are related to the conjugacy relation of [2].

3.5. PROPOSITION. *Let* $(\varphi,\tau):(X,D,\pi)\to(Y,E,\rho)$ *be a morphism such that* $\varphi: X \to Y$ *is a homeomorphism. Then there exists a commuting diagram of morphisms*



where (φ,ι) *is an equivariant isomorphism. Moreover,* $(1_X,\tau)$ *is an isomorphism iff* (φ,τ) *is an isomorphism.*

PROOF. Since $\varphi\colon X \to Y$ is a homeomorphism, F and σ can unambiguously be defined by

$$J_\sigma(x) := J_\rho(\varphi(x)) \quad \text{for } x \in X,$$
$$F := \bigcup\{\{x\} \times J_\sigma(x) \;\vdots\; x \in X\} \text{ and}$$
$$\sigma(x,t) := \varphi^{-1}(\rho(\varphi(x),t)) \text{ for } (x,t)\in F.$$

It is not hard to show that (X,F,σ) is a local dynamical system. Condition 2.2.4, for example, is verified as follows. For $(x,t)\in F$ we have, in view of the above definitions and condition 2.2.4 for the system (Y,E,ρ):

$$J_\sigma(\sigma(x,t)) = J_\rho(\varphi(\sigma(x,t))) = J_\rho(\rho(\varphi(x),t)) = J_\rho(\varphi(x))-t = J_\sigma(x)-t.$$

The remaining statements in the proposition can be verified without difficulty. □

In the sequel we need the following lemma.

3.6. LEMMA. *Suppose* $(\varphi,\tau)\colon (X,D,\pi)\to(Y,E,\rho)$ *satisfies the conditions of a morphism* (3.1) *with the possible exception of the continuity of* τ. *If* τ_z *is continuous for each* $z \in X$, *then for each* $x \in X$, *such that* $\varphi(x)$ *is not a rest point, and for all* $s,t \in \mathbb{R}$, *such that* t *and* $t+s \in J_\pi(x)$, *we have the following relation*

$$\tau_x(t+s) = \tau_x(t) + \tau_{\pi(x,t)}(s)$$

(cocycle relation).

PROOF. As $\varphi(\pi(x,t+s)) = \varphi(\pi(\pi(x,t),s))$, a straightforward calculation shows that for each $x \in X$ and for all s,t with $s,s+t \in J_\pi(x)$ we have

$$\rho_{\varphi(x)}(\tau_x(t+s)) = \rho_{\varphi(x)}(\tau_x(t)+\tau_{\pi(x,t)}(s)).$$

Consequently, $\tau_x(t+s)-\tau_x(t)-\tau_{\pi(x,t)}(s)$ is a period of $\varphi(x)$. So in case $\varphi(x)$ is not periodic the result follows. In case $\varphi(x)$ is periodic, for fixed x and t the function $s \mapsto \tau_x(t+s)-\tau_x(t)-\tau_{\pi(x,t)}(s)$ is continuous and equals 0 for $s = 0$. As the image of $J(x)-t$ under this mapping is discrete because $\varphi(x)$ is not a rest point, the result follows. □

4. INVARIANTS UNDER MORPHISMS

As for the invariants among the various properties of orbits in dynamical systems, the results can be summarized as follows. Orbits are mapped into orbits by morphisms, and positive and negative limit sets are mapped into positive and negative limit sets respectively. It follows that all those properties are invariant under morphisms which can be characterized by orbits and limit sets only (e.g. positive Poisson stability). Properties in the definition of which a certain distribution of the time parameter is involved (e.g. almost periodicity or recurrence) are not preserved by general morphisms, even not by isomorphisms. These properties are in general preserved by equivariant morphisms only.

4.1. <u>PROPOSITION</u>. *Let* $(\varphi,\tau):(X,D,\pi)\to(Y,E,\rho)$ *be a morphism of local dynamical systems. For every* $x \in X$ *we have:*

(i) $\varphi(\Gamma_\pi^+(x)) \subset \Gamma_\rho^+(\varphi(x))$, $\varphi(\Gamma_\pi^-(x)) \subset \Gamma_\rho^-(\varphi(x))$ *and* $\varphi(\Gamma_\pi(x)) \subset \Gamma_\rho(\varphi(x))$.

(ii) *If* x *is a rest point, then* $\varphi(x)$ *is a rest point.*

(iii) *If* x *is a periodic point, then* $\varphi(x)$ *is a periodic point.*

<u>PROOF</u>.

(i) Let $y \in \Gamma_\pi^+(x)$. Then for some $s > 0$ we have $y = \pi(x,s)$. By 3.1.3 we have $\varphi(y) = \varphi(\pi(x,s)) = \rho(\varphi(x),\tau(x,s))$. As τ_x is strictly increasing and $\tau_x(0) = 0$, it follows that $\tau(x,s) > 0$ and $\varphi(y) \in \Gamma_\rho^+(\varphi(x))$. The other statements in (i) are proved in a similar way.

(ii) If x is a rest point, then $\pi(x,t) = x$ for all t. It follows that $\varphi(x) = \rho(\varphi(x),\tau(x,t))$ for all t. In view of 3.1.2 we have $\varphi(x) = \rho(\varphi(x),s)$ for all s from some open interval containing 0. Hence $\varphi(x)$ is a rest point.

(iii) Let p denote the period of x. As $\pi(x,p) = x$, we have $\rho(\varphi(x),\tau(x,p)) = \varphi(x)$. Since τ_x is strictly increasing and $\tau(x,0) = 0$, we have $\tau(x,p) > 0$. It follows that $\varphi(x)$ is periodic. □

Next we show that limit sets are preserved by morphisms. The proof is based on the following lemma.

4.2. LEMMA. *Let* $(\varphi,\tau):(X,D,\pi)\to(Y,E,\rho)$ *be a morphism and let* $x \in X$. *If* $\Omega_\pi(x) \neq \phi$ *and* $\varphi(x)$ *is not a rest point, then* $J_\pi^+(x) = \mathbb{R}^+$ *and* $\tau_x(\mathbb{R}^+) = \mathbb{R}^+$. *In particular* $J_\rho^+(\varphi(x)) = \mathbb{R}^+$.

PROOF. Because $\Omega_\pi(x) \neq \phi$, we have $J_\pi^+(x) = \mathbb{R}^+$, as can be found in [10, Proposition 1]. Let $y \in \Omega_\pi(x)$ and let $r \in J_\pi(y)$ and $r > 0$. As D is open and τ is continuous, there is a neighbourhood U of y such that $U \times [0,r] \subset D$ and $\tau(z,r) > \frac{1}{2}\tau(y,r) > 0$ for every $z \in U$. Now there exists a sequence (t_n) in $J_\pi^+(x) = \mathbb{R}^+$ such that $t_{n+1} \geq t_n + r$ and $\pi(x,t_n) \in U$ for every $n \in \mathbb{N}$. Observe that

$$\tau(\pi(x,t_n),t_{n+1}-t_n) \geq \tau(\pi(x,t_n),r) \geq \tfrac{1}{2}\tau(y,r)$$

for every n, because of the strict monotonicity of the mapping $s \mapsto \tau(\pi(x,t_n),s)$. Now the cocycle relation for τ_x (3.6) implies that for every $n \in \mathbb{N}$

$$\tau(x,t_{n+1}) \geq \tau(x,t_n) + \tfrac{1}{2}\tau(y,r), \text{ whence}$$

$$\tau(x,t_{n+1}) \geq \tau(x,t_1) + \tfrac{1}{2}n\tau(y,r).$$

It follows that the interval $\tau_x(\mathbb{R}^+)$ contains arbitrarily large positive numbers, so $\tau_x(\mathbb{R}^+) = \mathbb{R}^+$. □

4.3. THEOREM. *Let* $(\varphi,\tau):(X,D,\pi)\to(Y,E,\rho)$ *be a morphism. For every* $x \in X$ *we have* $\varphi(\Omega_\pi(x)) \subset \Omega_\rho(\varphi(x))$ *and* $\varphi(A_\pi(x)) \subset A_\rho(\varphi(x))$.

PROOF. If $\varphi(x)$ is a rest point in (Y,E,ρ), then $\{\varphi(x)\} = \Gamma_\rho(\varphi(x)) = \Omega_\rho(\varphi(x))$. In view of the continuity of φ and because of 4.1 (i), we have $\varphi(\Omega_\pi(x)) \subset \varphi(\mathrm{cl}_X\, \Gamma_\pi(x)) \subset \mathrm{cl}_Y(\varphi(\Gamma_\pi(x)) \subset \mathrm{cl}_Y\, \Gamma_\rho(\varphi(x)) = \{\varphi(x)\}$ and the theorem follows. If $\varphi(x)$ is not a rest point and $\Omega_\pi(x) = \phi$, the theorem is obvious, so we assume that $\Omega_\pi(x) \neq \phi$. In this case, the theorem is proved as follows. First, observe that $\Omega_\rho(\varphi(x)) = \cap\{\mathrm{cl}_Y \rho_{\varphi(x)}[c_n,\infty) : n \in \mathbb{N}\}$ for any increasing sequence (c_n) such that $c_n \to \infty$. As τ_x induces a strictly increasing homeomorphism of $\mathbb{R}^+$ onto $\mathbb{R}^+$ in view of the lemma above, we get

$$\Omega_\rho(\varphi(x)) = \bigcap\{cl_Y(\rho_{\varphi(x)}[\tau_x(n),\infty)) : n \in \mathbb{N}\}$$
$$= \bigcap\{cl_Y(\varphi(\pi_x[n,\infty))) : n \in \mathbb{N}\}$$
$$\supset \bigcap\{\varphi(cl_X(\pi_x[n,\infty))) : n \in \mathbb{N}\}$$
$$\supset \varphi(\bigcap\{cl_X(\pi_x[n,\infty)) : n \in \mathbb{N}\}) = \varphi(\Omega_\pi(x)).$$

The proof that $\varphi(A_\pi(x)) \subset A_\rho(\varphi(x))$ can be carried out in a similar way or it can be based on the result just obtained and 2.3.3 and 3.4.4. □

We now collect some corollaries of the foregoing theorem. In view of the examples 2.3.3 and 3.4.4 we only discuss results about positive limit sets.

4.4. COROLLARY. *Let* $(\varphi,\tau):(X,D,\pi)\to(Y,E,\rho)$ *be a morphism. If* $x \in X$ *is positively Poisson-stable, then so is* $\varphi(x)$. *If* $x \in X$ *is positively Lagrange-stable, then so is* $\varphi(x)$, *and* $\varphi(cl_X(\Gamma^+_\pi(x))) = cl_Y(\Gamma^+_\rho(\varphi(x)))$.

PROOF. By definition x is positively Poisson-stable iff $x \in \Omega_\pi(x)$. By theorem 4.3, $\varphi(x) \in \varphi(\Omega_\pi(x)) \subset \Omega_\rho(\varphi(x)$. Hence $\varphi(x)$ is positively Poisson-stable. According to the definition, x is positively Lagrange-stable iff $cl_X(\Gamma^+_\pi(x))$ is compact. Then $\Omega_\pi(x) \neq \emptyset$ and $J^+_\pi(x) = \mathbb{R}^+$ by [10, Proposition 1]. As the case that $\varphi(x)$ is a rest point is almost trivial, we assume that $\varphi(x)$ is not a rest point. Then by lemma 4.2 we have $\tau_x(\mathbb{R}^+) = \mathbb{R}^+$ and similar to 4.1 (i) we get $\varphi(\Gamma^+_\pi(x)) = \Gamma^+_\rho(\varphi(x))$. Since $\varphi(cl_X(\Gamma^+_\pi(x))$ is compact, it is closed, whence $cl_Y(\Gamma^+_\rho(\varphi(x))) = \varphi(cl_X(\Gamma^+_\pi(x))$. It follows that $\varphi(x)$ is positively Lagrange-stable. □

4.5. Here are two examples which may illustrate the special role of condition 3.1.2 in the definition of morphism.

4.5.1. Let a local dynamical system (X,D,π) be given. Let $(X,X\times\mathbb{R},\rho)$ denote the trivial system on X (2.3.2). Define $\tau: X \times \mathbb{R} \to \mathbb{R}$ by $\tau(x,t) := 0$. The pair $(1_X,\tau)$ satisfies all the conditions in (3.1) of a morphism from $(X,X\times\mathbb{R},\rho)$ to (X,D,π) except for τ_x not being strictly increasing. It is clear that the pair $(1_X,\tau)$ does in general not preserve rest points.

4.5.2. Let $(S^1,S^1\times\mathbb{R},\pi)$ denote the system of rotations of the unit circle in the complex plane defined by $\pi(z,t) = ze^{it}$. Let $(\mathbb{R},\mathbb{R}\times\mathbb{R},\rho)$ be the system of translations of the real line defined by $\rho(x,t) := x + t$. The mappings $\varphi: S^1 \to \mathbb{R}$ and $\tau: S^1\times\mathbb{R}\to\mathbb{R}$ are defined by $\varphi(z) :=$ Re z and

$\tau(z,t) := (\mathrm{Re}\, z)((\cos t)-1) - (\mathrm{Im}\, z)(\sin t)$. The pair (φ,τ) satisfies all the conditions in (3.1) of a morphism from $(S^1, S^1\times\mathbb{R}, \pi)$ to $(\mathbb{R}, \mathbb{R}\times\mathbb{R}, \rho)$ except for τ_z not being increasing, although τ_z is increasing in a neighbourhood of 0 if $\mathrm{Im}\, z < 0$. The pair (φ,τ) does not preserve periodicity, limit sets, Poisson-stability and Lagrange-stability.

4.6. Properties involving a certain distribution of the time parameter are in general not preserved by morphisms. As an example we discuss the behaviour of the property of *recurrence*. Recall that a point x of a local dynamical system (X,D,π) is called *recurrent* if $J_\pi(x) = \mathbb{R}$ and for every neighbourhood U of x there exists a compact interval K in $\mathbb{R}$ such that $\mathbb{R} = K + A(x,U)$, where $A(x,U) := \{t \in \mathbb{R} : \pi(x,t) \in U\}$. Thus, x is recurrent iff for every neighbourhood U of x the gaps in $A(x,U)$ are bounded.

A recurrent point is positively and negatively Poisson-stable. Moreover, BIRKHOFF's recurrence theorem states that a point x is recurrent and positively and negatively Langrange-stable iff the orbit closure $\mathrm{cl}_X\, \Gamma_\pi(x)$ is a compact minimal set. See [7] or [8, pp.90-92] for more details.

First, we shall present some positive results about preservation of recurrence.

4.7. Let $(\varphi,\tau):(X,D,\pi)\to(Y,E,\rho)$ be a morphism, and let $x \in X$ be a recurrent point. For every neighbourhood V of $\varphi(x)$ in Y we have

$$\tau_x(A(x,\varphi^{-1}(V))) \subset A(\varphi(x),V).$$

Using this, it is quite obvious that if (φ,τ) is an equivariant morphism, then $\varphi(x)$ is recurrent.

More generally, if there exist $\varepsilon > 0$ and $\delta > 0$ such that $\tau_x(s)-\tau_x(t)\leq\varepsilon$ for all $s,t \in \mathbb{R}$ with $t \leq s \leq t+\delta$, then $\tau_x(A(x,\varphi^{-1}(V)))$ has bounded gaps, because $A(x,\varphi^{-1}(V))$ has so. Hence $A(\varphi(x),V)$ has bounded gaps, and $\varphi(x)$ is recurrent. Note, that this situation occurs when τ_x is uniformly continuous. In particular, $\varphi(x)$ is recurrent if x is recurrent and τ is given as in 3.4.2 with the function h bounded away from 0 on X.

4.8. <u>PROPOSITION</u>. *Let* $(\varphi,\tau):(X,D,\pi)\to(Y,E,\rho)$ *be a morphism. If* $x \in X$ *is reccurrent and positively and negatively Lagrange-stable, then so is* $\varphi(x)$.

PROOF. In view of corollary 4.4 we have $\varphi(cl_X \Gamma_\pi(x)) = cl_Y(\Gamma_\rho(\varphi(x)))$. By BIRKHOFF's recurrence theorem $cl_X \Gamma_\pi(x)$ is a compact minimal set. As the inverse image of an invariant subset of Y is an invariant subset of X in view of proposition 4.1 (i), the set $cl_Y(\Gamma_\rho(\varphi(x)))$ is compact and minimal. Again by BIRKHOFF's theorem the result follows. □

4.9. We now show by means of an example that the property of being a recurrent point need not be preserved by isomorphisms. The example is based on the well-known system of the irrational flow on the torus. Let T be the two-dimensional torus and let α be an irrational number. On T we shall define two systems. Using coordinates x and y, with $0 \le x < 1$ and $0 \le y < 1$, the first system is defined by the differential equation

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = \alpha.$$

The induced dynamical system (2.3.1), which turns out to be a global system, since T is compact, is denoted by $(T, T\times\mathbb{R}, \pi)$. It is well-known that every point z of T is recurrent. Indeed $cl_T(\Gamma_\pi(z)) = T$ for every $z \in T$.

Let p be a point of T and let $h\colon T \to \mathbb{R}$ be a continuous mapping such that $h(p) = 0$ and $h(z) > 0$ for every $z \in T\setminus\{p\}$ (e.g. h(z) equals the distance of z to p). The second system $(T, T\times\mathbb{R}, \rho)$ on T is defined by the differential equations

$$\frac{dx}{dt} = h(x,y) \quad \text{and} \quad \frac{dy}{dt} = \alpha\, h(x,y).$$

In $(T, T\times\mathbb{R}, \rho)$ the point p is a rest point and T is not a minimal set. And for every $z \in T$ we have that if $p \notin \Gamma_\pi(z)$, then $\Gamma_\pi(z) = \Gamma_\rho(z)$. Consequently, $cl_T(\Gamma_\rho(z)) = T$, whenever $p \notin \Gamma_\pi(z)$. As T is not minimal in $(T, T\times\mathbb{R}, \rho)$, BIRKHOFF's recurrence theorem implies that z is not recurrent in $(T, T\times\mathbb{R}, \rho)$ whenver $p \notin \Gamma_\pi(z)$. The set $T\setminus\{p\}$ is denoted by X. We consider the restrictions of both systems to X (3.4.3). The restriction of the first system is denoted by (X, D, π_X) and the restriction of the second system, which turns out to be a global system, as is easily seen, is denoted by $(X, X\times\mathbb{R}, \rho_X)$. Observe that the systems thus obtained are isomorphic (3.4.2). However, any point z in the dense G_δ-subset $T\setminus\Gamma_\pi(p)$ of X is recurrent in (X, D, π_E) but not recurrent in $(X, X\times\mathbb{R}, \rho_E)$.

4.10. To conclude the discussion on invariants, we show that Liapunov stability of rest points is in general not preserved by morphisms, but is preserved by isomorphisms. A rest point x of a local dynamical system (X,D,π) is called *Liapunov-stable* if for every neighbourhood U of x there is a neighbourhood V of x such that for every $y \in V$ we have $\Gamma_\pi^+(y) \in U$ (cf. [7]). One remark should be made here. Usually it is also required that $J_\pi^+(y) = \mathbb{R}^+$ for every $y \in V$. That condition however is redundant, as can be seen as follows.

Suppose that x is a rest point which satisfies the above definition of Liapunov stability. As x is a rest point, $J_\pi^+(x) = \mathbb{R}^+$. Since D is an open set (2.2.0), there is a neighbourhood U of x such that $1 \in J_\pi^+(y)$ for every $y \in U$. According to the definition of Liapunov stability there exists a neighbourhood W of x such that $\Gamma_\pi^+(y) \subset U$ for every $y \in W$. Our claim is that $\mathbb{R}^+ = J_\pi^+(y)$ for every $y \in W$. Assuming that $\omega_\pi(y_o) < \infty$ for some $y_o \in W$, let (t_n) be a strictly increasing sequence such that $t_n \to \omega_\pi(y_o)$. Then $\omega_\pi(\pi(y_o,t_n)) \to 0$ and $\omega_\pi(\pi(y_o,t_{n_o})) < 1$ for some n_o. As $\pi(y_o,t_{n_o}) \in U$, we have a contradiction.

The following result easily follows from proposition 4.1 (i).

4.11. PROPOSITION. *Let* $(\varphi,\tau):(X,D,\pi)\to(Y,E,\rho)$ *be an isomorphism. If* x *is a Liapunov-stable rest point, then so is* $\varphi(x)$. □

4.12. Here is an example showing that Liapunov stability of rest points need not be preserved by (equivariant) morphisms. Consider in $\mathbb{R}$ the differential equation $\frac{dx}{dt} = x(1-x)$. $X := (0,1]$ is an invariant subset of the induced dynamical system (2.3.1). The restriction of this system to X is a global system and is denoted by $(X,X\times\mathbb{R},\rho)$. The rest point 1 is Liapunov-stable. The map $h: X \to S^1$ is defined by $h(x) := e^{2\pi ix}$. On S^1 a global dynamical system is defined by requiring that (h,ι) be a equivariant morphism. In the system so obtained $h(1)$ is a rest point, but is not Liapunov-stable.

5. ORBIT-PRESERVING MAPS

The isomorphism type used in [7] is the geometric equivalence, i.e. a phase space homeomorphism preserving orbits. As is shown in [10] there is a close relationship between geometric equivalence and isomorphism as introduced by URA and the isomorphisms discussed in this paper. The obstructions to a complete coincidence of the notions of isomorphism are mainly caused by rest points and change of orientation. See [5] and [10] for a detailed discussion. In this section we extend the above mentioned relationship to one dealing with morphisms. Of course, the obstructions cannot be removed.

5.1. An *arc* in X is the image of a topological embedding $h: [0,1] \to X$ and is denoted by $\widehat{ab}$ or $\widehat{axb}$ where $a := h(0)$, $b := h(1)$ and $x := h(t)$ for some $t \in (0,1)$. Let (X,D,π) and (Y,E,ρ) be local dynamical systems. A mapping $\varphi: X \to Y$ is said to be *orbit-preserving* if the following conditions are satisfied:

5.1.1. φ is continuous.

5.1.2. $\varphi(\Gamma_\pi(x)) \subset \Gamma_\rho(\varphi(x))$ for every $x \in X$.

5.1.3. For every $x \in X$ such that $\varphi(x)$ is not a rest point in (Y,E,ρ) there exists an arc $\widehat{axb}$ such that $\widehat{axb} \subset \Gamma_\pi(x)$ and $\varphi|_{\widehat{axb}}$, the restriction of φ to $\widehat{axb}$, is injective.

The main purpose of this section is to construct morphisms from orbit-preserving mappings. First it is shown that a morphism induces an orbit-preserving mapping.

5.2. PROPOSITION. *If* $(\varphi,\tau): (X,D,\pi) \to (Y,E,\rho)$ *is a morphism, then* φ *is orbit preserving.*

PROOF. That conditions 5.1.1 and 5.1.2 are satisfied follows from 3.1.1 and 4.1 (i) respectively. Suppose that $x \in X$ and that $\varphi(x)$ is not a rest point. Choose $s > 0$ in such a way that 2s is less than the period of $\varphi(x)$. As τ_x is continuous, there is a $\delta > 0$ such that $[-\delta,\delta] \subset J_\pi(x)$ and $\tau_x[-\delta,\delta] \subset (-s,s)$. Put $\alpha := \tau_x(-\delta)$ and $\beta := \tau_x(\delta)$. In view of condition 3.1.2 we have $\alpha < 0 < \beta$. Observe that $\beta-\alpha$ is less than the period of $\varphi(x)$. Consequently $\rho_{\varphi(x)}|_{[\alpha,\beta]}$ is injective and by 3.1.2 we have that $\rho_{\varphi(x)}\circ\tau_x|_{[-\delta,\delta]}$ is injective. As $\varphi\circ\pi_x = \rho_{\varphi(x)}\circ\tau_x$ by 3.1.3, it follows that $\varphi\circ\pi_x|_{[-\delta,\delta]}$ is injective. Hence $\varphi|_{\pi_x[-\delta,\delta]}$ is injective. Putting $a := \pi_x(-\delta)$ and $b := \pi_x(\delta)$ we get the arc $\widehat{axb}$ satisfying 5.1.3. $\square$

An important tool in the proof of the main result is the following theorem due to the J. & M. LEWIN [2, Theorem 1.25]. In [2] the theorem is stated and proved for global dynamical systems. The modifaction of the proof for local dynamical systems is straightforward.

5.3. THEOREM. *Let* (X,D,π) *be a local dynamical system and suppose that* x *is not a periodic point. If* C *is a compact and connected subset of* $\Gamma_\pi(x)$, *then* $\pi_x^{-1}(C)$ *is a compact (possibly degenerated) interval* J *and* $\pi_x|_J\colon J \to C$ *is a homeomorphism.* □

5.4. PROPOSITION. *Let* (X,D,π) *and* (Y,E,ρ) *be local dynamical systems. Suppose* $\varphi\colon X \to Y$ *is an orbit-preserving map. For every* $x \in X$ *we have*

(i) *If* x *is a rest point, then so is* $\varphi(x)$.

(ii) *If* x *is a periodic point, then so is* $\varphi(x)$.

PROOF. If $\varphi(x)$ is not a rest point, then according to 5.1.3 there exists an arc $\widehat{axb}$ in $\Gamma_\pi(x)$ such that $\varphi|_{\widehat{axb}}$ is injective. In particular $\Gamma_\pi(x)$ consists of more than one point and x cannot be a rest point. This proves (i).

To prove (ii) assume that x is periodic and $\varphi(x)$ is not. As $\Gamma_\pi(x)$ is compact and connected, in view of 5.1.2 the set $C := \varphi(\Gamma_\pi(x))$ is a compact and connected subset of $\Gamma_\rho(\varphi(x))$. By Theorem 5.3 we have that C is homeomorphic to a compact interval $J := [a,b] \subset J_\rho(\varphi(x))$ under the homeomorphism $(\rho_{\varphi(x)}|_J)^{-1}$. Choose $p \in \Gamma_\pi(x)$ such that $\varphi(p) = \rho_{\varphi(x)}(a)$. By condition 5.1.3 there is an arc $\widehat{cpd}$ in $\Gamma_\pi(x)$ such that $\varphi|_{\widehat{cpd}}$ is injective, hence a homeomorphic embedding of $\widehat{cpd}$ in C. It follows that $(\rho_{\varphi(x)}|_J)^{-1}(\varphi(\widehat{cpd}))$ is a compact interval which is contained in J and which has the point a as an internal point. This is a contradiction. □

The following lemma is the first step of the construction of a morphism from an orbit-preserving map.

5.5. LEMMA. *Let* (X,D,π) *and* (Y,E,ρ) *be local dynamical systems and let* $\varphi\colon X \to Y$ *be an orbit-preserving map. Then for every* $x \in X$ *such that* $\varphi(x)$ *is not a rest-point, there exists a unique continuous mapping* $\tau_x\colon J_\pi(x) \to J_\rho(\varphi(x))$ *satisfying the following conditions:*

(i) $\tau_x(0) = 0$,

(ii) τ_x *is strictly monotone.*
(iii) $\varphi(\pi(x,t)) = \rho(\varphi(x),\tau_x(t))$ *for every* $t \in J_\pi(x)$.

PROOF. We shall first show the existence of a unique continuous mapping τ_x: $J_\pi(x) \to J_\rho(\varphi(x))$ satisfying the conditions (i) and (iii). To do so, we first assume that $\varphi(x)$ is not a periodic point. As under this assumption $\rho_{\varphi(x)}: J_\rho(\varphi(x)) \to \Gamma_\rho(\varphi(x))$ is bijective, the mapping τ_x must be defined by $\tau_x := \rho^{-1}_{\varphi(x)} \circ \varphi \circ \pi_x$, in view of condition (iii). This proves existence and unicity of τ_x for this case, and it is clear that conditions (i) and (iii) are satisfied. The continuity of τ_x is proved as follows. Let I be any compact interval in $J_\pi(x)$. As $\varphi(\pi_x(I))$ is a compact and connected subset of $\Gamma_\rho(\varphi(x))$, condition (iii) and Theorem 5.3 imply that $J := \tau_x(I)$ is a compact interval in $J_\rho(\varphi(x))$ and that $\rho_{\varphi(x)}|_J$ is a homeomorphism. Since $\varphi\circ\pi_x = \rho_{\varphi(x)} \circ \tau_x$ this implies that $\tau_x|_I$ is continuous. The continuity of τ_x on $J_\pi(x)$ now easily follows.

Next we consider the case that $\varphi(x)$ is a periodic point but not a rest point. In this case $J_\rho(\varphi(x)) = \mathbb{R}$ and $\Gamma_\rho(\varphi(x))$ is homeomorphic to the unit circle S^1. The mapping $\rho_{\varphi(x)}: \mathbb{R} \to \Gamma_\rho(\varphi(x))$ is a covering map, and the mapping $\varphi\circ\pi_x: J_\pi(x) \to \Gamma_\rho(\varphi(x))$ can be lifted to a unique continuous mapping $\tau_x: J(x) \to \mathbb{R} = J_\rho(\varphi(x))$ which satisfies (i) and (iii) (cf. [4], p.139). This concludes the proof of existence and unicity of a continuous mapping $\tau_x: J_\pi(x) \to J_\rho(\varphi(x))$ satisfying the conditions (i) and (iii). In order to show that such a mapping also satisfies condition (ii) it is sufficient to prove that τ_x is locally injective, since τ_x is a continuous mapping of an interval into an interval.

Let $t \in J_\pi(x)$ and $c := \pi_x(t)$. As φ is orbit-preserving there is an arc $\widehat{acb}$ in $\Gamma_\pi(c) = \Gamma_\pi(x)$ such that $\varphi|_{acb}$ is injective. Because, by proposition 5.4, the point x is not a rest point, there is an interval $I \subset J_\pi(x)$ which is a neighbourhood of t such that $\pi_x|_I$ is an injective mapping of I into $\widehat{acb}$; indeed, if x is not periodic, this follows from theorem 5.3 with $C = \widehat{acb}$, and if x is periodic this follows from the fact that π_x is in that case a covering map of $\mathbb{R}$ onto $\Gamma_\pi(x)$. Now $\varphi\circ\pi_x|_I$ is injective, and as $\varphi\circ\pi_x = \rho_{\varphi(x)} \circ \tau_x$, it follows that $\tau_x|_I$ is injective. This completes the proof that τ_x satisfies condition (ii). □

The following theorem is the main result of this section.

5.6. THEOREM. *Let* (X,D,π) *and* (Y,E,ρ) *be local dynamical systems. Let* G *be the set of rest points of* Y. *Suppose* $\varphi: X \to Y$ *is orbit preserving. Then there exists a unique mapping* $\tau: D\backslash(\varphi^{-1}(G) \times \mathbb{R}) \to \mathbb{R}$ *such that the following conditions are satisfied.*

(i) *For every* $x \in X\backslash\varphi^{-1}(G)$ *the mapping* $\tau_x: J_\pi(x) \to \mathbb{R}$ *defined by* $\tau_x(t) := \tau(x,t)$, *is strictly monotone and satisfies* $\tau_x(0) = 0$ *and* $\tau_x(J_\pi(x)) \subset J_\rho(\varphi(x))$.

(ii) *For every* $(x,t) \in D\backslash(\varphi^{-1}(G)\times\mathbb{R})$ *we have* $\varphi(\pi(x,t)) = \rho(\varphi(x),\tau(x,t))$.

(iii) τ *is continuous.*

PROOF. According to the preceding lemma, for every $x \in X\backslash\varphi^{-1}(G)$ there is a unique continuous mapping $\tau_x: J_\pi(x) \to J_\rho(\varphi(x))$ satisfying conditions (i), (ii) and (iii) of 5.5. Define $\tau: D\backslash(\varphi^{-1}(G) \times \mathbb{R}) \to \mathbb{R}$ by the rule $\tau(x,t) := \tau_x(t)$. It only remains to show that τ is continuous. Let $(x,t) \in D\backslash(\varphi^{-1}(G) \times \mathbb{R})$. Assume that $t \geq 0$ and that t_x is strictly increasing. In the other cases the proof is similar. Let $s := \tau(x,t)$ and $y := \varphi(x)$. Let $\varepsilon > 0$ be given. We shall exhibit a neighbourhood W of x and a $\delta > 0$ such that for all $(x',t') \in W \times (t-\delta,t+\delta)$ we have $|\tau(x',t')-s| < \varepsilon$. As this final part of the proof is similar to the proof of lemma 4 in section 3 of [10], we shall skip the details. In case $y = \varphi(x)$ is a periodic point, we first assume that s is less than the period of y. Then we also may assume that $s + 2\varepsilon$ is less than the period of y. Thus the mapping $\rho_y|_{[-\varepsilon,s+\varepsilon]}$ is a topological embedding of $[-\varepsilon,s+\varepsilon]$ in Y. In Y there can be chosen disjoint neighbourhoods U_1, U_2 and U_3 of the points $\rho(y,-\varepsilon)$, $\rho(y,s+\varepsilon)$ and the compact set $\rho(\{y\} \times [0,s])$ respectively. Also disjoint neighbourhoods U_4 and U_5 of the compact set $\rho(\{y\} \times [-\varepsilon,s-\varepsilon])$ and the point $\rho(y,s)$ respectively can be chosen. Because ρ is continuous, there is a neighbourhood V of y such that $\rho(V\times\{-\varepsilon\}) \subset U_1$, $\rho(V\times\{s+\varepsilon\}) \subset U_2$ and $\rho(V\times[-\varepsilon,s-\varepsilon]) \subset U_4$. As φ is continuous there is a neighbourhood W_1 of x such that $\varphi(W_1) \subset V$. Because $\varphi\circ\pi: D \to Y$ is continuous and because $\varphi\circ\pi(\{x\} \times [0,t]) = \rho(\{y\} \times [0,s])$, as follows from condition (ii) and the monotonicity of τ_x, there are a neighbourhood W_2 of x and a $\delta_1 > 0$ such that $\varphi\circ\pi(W_2 \times [-\delta_1,t+\delta_1]) \subset U_3$. As $\varphi(\pi(x,t)) = \rho(y,s)$, there are a neighbourhood W_3 of x and a $\delta_2 > 0$ such that $\varphi\circ\pi(W_3 \times [t-\delta_2,t+\delta_2]) \subset U_5$. Now let $W := W_1 \cap W_2 \cap W_3$ and $\delta := \min\{\delta_1,\delta_2\}$.

Let $(x',t') \in (W \times [t-\delta,t+\delta]) \cap (D\backslash(\varphi^{-1}(G)\times\mathbb{R}))$; we shall show that $|\tau(x',t') - s| < \varepsilon$. Let $I := [0,t']$ in case $t' \geq 0$ and $I := [t',0]$ in case $t' < 0$. Now $I \subset [-\delta,t+\delta]$ and $\{x'\} \times I \subset W_2 \times [-\delta_1,t+\delta_1]$ and we have $\rho(\{\varphi(x')\} \times \tau_{x'}(I)) = \varphi\circ\pi(\{x'\} \times I) \subset U_3$. We also have $\rho(\varphi(x'),-\varepsilon) \in \rho(V \times \{-\varepsilon\}) \subset U_1$ and $\rho(\varphi'(x),s+\varepsilon) \subset U_2$. Because U_1, U_2 and U_3 are pairwise disjoint and because $\tau_{x'}$ is continuous, it follows that $\tau_{x'}(I)$ is an interval which does not contain the points $-\varepsilon$ and $s+\varepsilon$. It follows that $\tau_{x'}(I) \subset (-\varepsilon,s+\varepsilon)$, since $0 \in \tau_{x'}(I)$. In particular, $\tau(x',t') \in (-\varepsilon,s+\varepsilon)$. Now assuming $\tau(x',t') \in (-\varepsilon,s-\varepsilon)$, we get

$$\varphi(\pi(x',t')) = \rho(\varphi(x'),\tau(x',t')) \in \rho(V \times [-\varepsilon,s-\varepsilon]) \subset U_4$$

and also

$$\varphi(\pi(x',t')) \in \varphi(\pi(W_3 \times [t-\delta_2,t+\delta_2])) \subset U_5.$$

Because U_4 and U_5 are disjoint, we must have $\tau(x',t') \in (s-\varepsilon,s+\varepsilon)$. This proves the continuity of τ in (x,t). The restriction that s is less than the period of y can be removed in the same way as in [10], using lemma 3.6. □

Now we collect the corollaries of the preceding theorem. Let (X,D,π) be a local dynamical system. If X is not connected, every component of X is an invariant subset. The *restriction of* (X,D,π) *to a component* C i.e. the local dynamical system $(C,D\cap(X\times\mathbb{R}), \pi|_{D\cap(C\times\mathbb{R})})$ will be denoted by (C,D_C,π_C).

5.7. <u>THEOREM</u>. *Let* (X,D,π) *and* (Y,E,ρ) *be local dynamical systems. Let* G *be the set of rest point of* Y. *Suppose* $\varphi: X \to Y$ *is orbit-preserving. Then there exists a mapping* $\tau: D\backslash(\varphi^{-1}(G)\times\mathbb{R}) \to \mathbb{R}$ *such that for each component* C *of* $X\backslash\varphi^{-1}(G)$ *either*

$$(\varphi|_C,\tau|_{D_C}): (C,D_C,\pi_C) \to (Y,E,\rho)$$

or

$$(\varphi|_C,-\tau|_{D_C}): (C,D_C,\pi_C) \to (Y,E_*,\rho_*)$$

is a morphism. (Recall that (Y,E_*,ρ_*) *denotes the reverse system* (2.3.3)).

<u>PROOF</u>. Let τ be defined as in theorem 5.6. As τ is continuous, on each component C of $X\backslash\varphi^{-1}(G)$ the mappings τ_x are either increasing for every $x \in C$ or decreasing for every $x \in C$. □

The following theorem is a reformulation of a result obtained by URA [10], which has been mentioned in the introduction.

THEOREM. *Let* (X,D,π) *and* (Y,E,ρ) *be local dynamical systems. Suppose* $\varphi: X \to Y$ *is a homeomorphism such that for every* $x \in X$ *we have* $\varphi(\Gamma_\pi(x)) = \Gamma_\rho(\varphi(x))$. *Let* F *be the set of rest points of* X. *Then there exists a mapping* $\tau: D\backslash(F\times\mathbb{R}) \to \mathbb{R}$ *such that for each component* C *of* $X\backslash F$ *either*

$$(\varphi|_C, \tau|_{D_C}) : (C, D_C, \pi_C) \to (\varphi(C), E_{\varphi(C)}, \rho_{\varphi(C)})$$

or

$$(\varphi|_C, -\tau|_{D_C}) : (C, D_C, \pi_C) \to (\varphi(C), E_{\varphi(C)^*}, \rho_{(C)^*})$$

is an isomorphism.

PROOF. It is clear that φ is orbit-preserving. As rest points are characterized by the property that the orbit consists of a single point, it is easily seen that $G := \varphi(F)$ is the set of rest points in (Y,E,ρ). We also have that C is a component of $X\backslash F$ iff $\varphi(C)$ is a component of $Y\backslash G$. From the preceding theorem it now follows that either $(\varphi|_C, \tau_{D_C})$ or $(\varphi|_C, -\tau_{D_C})$ is a morphism. Observe that the spatial action $\varphi|_C: C \to \varphi(C)$ is a homeomorphism.
In [1, Theorem 4.5] it is proved that a morphism whose spatial action is a homeomorphism, is an isomorphism. □

REMARK. The last conclusion in the above proof cannot be drawn from proposition 3.3. What is essentially achieved in [1], is the removal of the condition that every τ_x is surjective (3.3 (iii)).

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